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THE HISTORY OF ZENO'S ARGUMENTS ON MOTION.

By FLORIAN CAJORI, Colorado College.

VI.

6. NEWTON, BERKELEY, JURIN, ROBINS AND OTHERS.

Whether certain variables can reach their limits or not is the vital issue in the "Achilles." For that reason Newton's statements on this point are of interest:

> ". . . those ultimate ratios with which quantities vanish are not truly the ratios of ultimate quantities, but limits toward which the ratios of quantities decreasing without limit do always converge; and to which they approach nearer than by any given difference, but never go beyond, nor in effect attain to, till the quantities are diminished in *infinitum*."[1]

That Newton let his variables reach their limits appears even more clearly in the following passage:

> "Quantities, and the ratios of quantities, which in any finite time converge continually to equality, and before the end of that time approach nearer the one to the other than by any given difference, become ultimately equal."[2]

Other passages in the first book of the *Principia* allow variables to reach their limits. While Newton's exposition is not as explicit as one might wish, nor free from objection, he deserves the credit of perceiving that variables may reach their limits and that variables arising in mechanics are usually of such a nature that they do reach their limits.

As is well known, the foundations of the calculus were severely attacked by Bishop Berkeley. His first published statement on this subject appears in his *Alciphron, or the Minute Philosopher* (1732), penned while he and his wife were sojourning at Newport, Rhode Island. He says that mathematical science falls short of "those clear and distinct ideas" which many "expect and insist upon in the mysteries of religion. . . . Such are those which have sprung up

[1] Newton's *Principia*, Book I, Section I, last scholium.
[2] Newton's *Principia*, Book I, Section I, Lemma I.

in geometry about the nature of the angle of contact, the doctrine of proportions, of indivisibles, infinitesimals, and divers other points." He argues that, just as mathematics, though involved in obscurities, is esteemed excellent and useful, so articles of Christian faith should be accepted as none the less true and excellent, because they afford matter of controversy. These ideas Berkeley elaborates more fully in his *Analyst* (1734) and the *Defence of Free-thinking in Mathematics* (1735). These attacks on the foundations of the calculus deserve mention in this history, even though no reference is made to Zeno's arguments. Berkeley was familiar with Zeno's arguments, for in an earlier essay he refers to them twice, though without critical comment.[1]

The *Analyst* is a discourse addressed to an unnamed infidel mathematician, said to have been Dr. Halley. Berkeley's lengthy discourse dwells mainly on two points: (1) The conception of fluxions is unintelligible, since they are the ratios of quantities that have no magnitudes, (2) the derivation of the fluxion of x^n rests on a violation of an axiomatic canon of sound reasoning. Newton did not take the fluxions infinitely small, but he originally took the *moments of fluxions* to be infinitely small. Later he discarded the infinitely little and explained fluxions by his theory of prime and ultimate ratios. Berkeley argued with great acuteness against the infinitely small. As his arguments do not directly apply to the "Dichotomy" and the "Achilles" we shall not go into details except to quote part of *Query* 21 in his *Analyst*, where he inquires, "whether the supposed infinite divisibility of finite extension hath not been a snare to mathematicians and a thorn in their sides?" Before this Berkeley had discussed this vital question quite fully in his *Principles of Human Knowledge*, first printed in 1710, then again in 1734. Infinite divisibility, because inapprehensible by our senses, is dismissed from his philosophy as void of meaning or involving contradictions. We quote the following:[2]

"And, as this notion is the source from whence do spring all those amusing geometrical paradoxes which have such a direct repugnancy to the plain common sense of mankind, and are admitted with so much reluctance into a mind not yet debauched by learning. . . . Of late speculations about Infinites have run so high, and grown to such strange notions, as have occasioned no small scruples and disputes among the geometers of the present age. Some there are of great note who, not content with holding that finite lines may be divided into an infinite number of parts, do yet farther maintain that each of those infinitesimals is itself subdivisible into an infinity of other parts or infinitesimals of a second order, and so on *ad infinitum*. These, I say, assert there are infinitesimals of infinitesimals of infinitesimals, etc., without ever coming to an end; so that according to them an inch does not barely contain an infinite number of parts, but an infinity of an infinity of an infinity *ad infinitum* of parts. Others there be who hold all orders of infinitesimals below the first to be nothing at all; . . . Have we not therefore reason to conclude they are *both* in the wrong, and that there is in effect no such thing as parts infinitely small, or an infinite number of parts contained in any finite quantity? . . . If it be said that several theorems undoubtedly true are discovered by methods in which infinitesimals are made use of, which could never have been if their existence included a contradiction in it—I answer that upon a thorough examination it will not be found that in any instance it is necessary to make use of or conceive infinitesimal parts of finite lines, or even quantities less than the *minimum sensible;*

[1] A. C. Fraser, *The Works of George Berkeley*, D.D., Vol. II, Oxford, 1871, p. 501, also Vol. III, pp. 76, 91.

[2] Berkeley, "Principles of Human Knowledge," *The Works of George Berkeley*, Vol. I, pp. 220, 223–225.

nay, it will be evident this is never done, it being impossible. And, whatever mathematicians may think of fluxions, or the differential calculus and the like, a little reflection will show them that, in working by those methods, they do not conceive or imagine lines or surfaces less than what are perceivable to sense."

The *Analyst* gave rise to a spirited discussion. An anonymous reply by Philalethes Cantabrigiensis appeared under the title, *Geometry, no Friend to Infidelity; or, A Defence of Sir Isaac Newton and the British Mathematicians*, London, 1734. The authorship of this letter has been attributed to Conyers Middleton and Robert Smith, but George A. Gibson makes it plain that the author is James Jurin,[1] a noted Cambridge physician and an admirer of Newtonian philosophy, which he had imbibed from Newton himself. Philalethes admits that the doctrine of fluxions is involved in difficulties, but claims that they are not insuperable. Gibson calls this reply "an extremely weak defence" of the doctrine of fluxions. In 1735 Berkeley published his *Defence, etc.*, alluded to above, to which Philalethes replied in a pamphlet entitled *The Minute Mathematician: or the Freethinker no just Thinker.* Berkeley did not make answer to this, nor to a publication of the same year by Benjamin Robins, a mathematician and military engineer, which appeared in London under the title: *A Discourse concerning the Nature and Certainty of Sir Isaac Newton's Methods of Fluxions and of Prime and Ultimate Ratios.* A controversy arose between Philalethes and Robins which bears more closely on our present topic than that between Philalethes and Berkeley. Robins and Philalethes differed in the interpretation of Newton; they "began that long struggle in which," as Gibson puts it, "Robins proved his immense superiority to his antagonist, alike in temper, in general mathematical learning, and in special knowledge of Newton's fluxionary methods." The part of the debate which interests us just now relates to the variable's reaching its limit. On this point Robins's vision was somewhat circumscribed; he held that no variable could possibly reach its limit. This interpretation of Newton is at variance with that usually accepted. For the purposes of debate it was no doubt easier for Robins to limit himself to variables which do not reach their limits; from the standpoint of mathematical theory which should be broad enough to explain all ordinary phenomena of motion, his position was unfortunate. Says Robins:

"It was urged that the quantities or ratios, asserted in this method to be ultimately equal, were frequently such as could never absolutely coincide. As, for instance, the parallelograms inscribed within the curve, in the second *lemma* of the first book of Sir Isaac Newton's *Principia*, cannot by any division be made equal to the curvilinear space they are inscribed in, whereas in that *lemma* it is asserted that they are ultimately equal to that space.

"Here two different methods of explanation have been given. The first, supposing that by ultimate equality a real assignable coincidence is intended, asserts that these parallelograms and the curvilinear space do become actually, perfectly, and absolutely equal to each other."

This last view described by Robins was the view of Jurin. No doubt Jurin followed more nearly in Newton's footsteps than did Robins. Newton declares

[1] G. A. Gibson, Review of Cantor's "Geschichte der Mathematik," Vol. 3, in *Proceedings of the Edinburgh Mathematical Society*, Vol. XVII, 1898–99. Gibson's article gives the most complete account of the *Analyst* controversy with which we are familiar.

that the variable becomes "ultimately equal" to its limit, yet Robins insists that he must have seen they would always remain unequal. Robins's contention was hardly valid; whether a variable reaches its limit or not depends wholly upon the variable. Now a law of variation may be artificially established by the human mind. That law may be such that the variable reaches its limit, or it may be such that the variable does not reach its limit. Apparently Robins, perhaps unconsciously, assumed laws of variation which kept the variable and its limit constantly apart, while the great Newton conceived modes of variability not limited to such conditions. How Robins came to insist that his views were those deduced from Newton's *Principia* is elucidated by him in the following passage, in which he says:

> "[Newton] has given such an interpretation of this method as did no ways require any such coincidence. In his explication of that doctrine of prime and ultimate ratios he defines the ultimate magnitude of any varying quantity to be the limit of that varying quantity which it can approach within any degree of nearness, and yet can never pass. And in like manner the ultimate ratio is the limit of that varying ratio."

The reader may compare this passage from Robins with the passages in Newton's *Principia* which we quoted earlier. That Philalethes failed to do Newton justice is clearly brought out by Gibson. According to Philalethes an ultimate ratio is not the *limit* of a varying ratio, but the last value of a ratio. Berkeley very properly argued that there is no last value of the augments except zero, so that the phrase "the ratio with which they vanish," used by Newton himself, does not represent any mathematical operation, and itself requires explanation. Gibson claims that Newton's terminology of first and last ratios was unfortunate, "as it lent itself too readily to an interpretation in the sense of indivisibles; and it was this interpretation that Berkeley and Philalethes alike proceeded upon. Were that interpretation correct, then Berkeley's contentions would in the main be fully justified."

We may sum up the discussion by saying that Berkeley did not directly inquire whether Achilles caught the tortoise or not; that according to the teachings of Newton and Jurin on limits, Achilles did catch the tortoise, though it is not quite evident how the feat was accomplished; that Robins's theory did not allow Achilles to overtake the tortoise, though Achilles would come tantalizingly near doing so.

It was in 1710 that Bayle's famous dictionary was translated from French into English. We are not able to trace any immediate influence of the article on "Zeno of Elea" upon English thought. In 1713 appeared the *Clavis Universalis* of the English divine, Arthur Collier, an idealist who aimed to prove in his book the non-existence of the external world. As his fifth argument he considers motion. He does not mention Zeno, nor any other philosopher, but deserves to rank among Zeno's boldest and most reckless disciples. A few quotations will suffice.[1]

> "A world, in which it is both possible and impossible that there should be any such thing as motion, is not at all;

[1] A. Collier, *Clavis Universalis*, edited by Ethel Bowman, Chicago, 1909, pp. 80–82.

But this is the case of an external world;
Ergo, there is no such world."

". . . Now in such translation the space or line through which the body moved is supposed to pass, must be actually divided into all its parts. This is supposed in the very idea of motion; but this all is infinite, and this infinite is absurd, and consequently it is equally so, that there should be any motion in an external world."

". . . to affirm that a line by motion or otherwise is divided into infinite parts, is in my opinion to say all the absurdities in the world at once. For, *first*, this supposes a number actually infinite, that is, a number to which no unit can be added, which is a number of which there is no sum total, that is, no number at all; consequently, *secondly*, by this means the shortest motion becomes equal to the longest, since a motion to which nothing can be added must needs be as long as possible. This also, *thirdly*, will make all motions equal in swiftness, it being impossible for the swiftest in any stated time to do more than pass through infinite points, which yet the shortest is supposed to do."

Collier gives no evidence of having looked into the higher mathematics as did Berkeley and Hume. After referring to the angle of contact between a circle and its tangent, which "is infinitely less than any rectilineal angle," Hume concludes "that all the ideas of quantity upon which mathematicians reason, are nothing but particular, and such as are suggested by the senses and imagination and consequently, cannot be infinitely divisible."[1]

Zeno's arguments appear to have been discussed but little in England during the second half of the eighteenth century. Charles Hutton refers to them in the article "Zeno, Eleates" in his *Mathematical and Philosophical Dictionary*, London, 1795. He describes the "Achilles" and remarks that "the fallacy will soon be detected," as the time can easily be computed when Achilles will not only have overtaken, but actually passed the tortoise.

Some English mathematicians kept in the path laid out by Newton, by teaching that a variable reaches its limit. Especially is this true of mathematicians at Cambridge, from Jurin to Whewell and from Whewell to Todhunter. Says Whewell:[2] "A magnitude is said to be *ultimately equal* to its Limit; and the two are said to be *ultimately in a ratio of equality.*"

Hutton, who was professor of mathematics at the Royal Military Academy, Woolwich, says, in his *Dictionary*, under the word "Limit," that the variable "can never go beyond it." A different exposition was given by Augustus De Morgan. In the article "Progressions" in the *Penny Cyclopædia* (London, 1841) he says of Achilles: "Let him go as far as he may, he must always come up to where the tortoise was before he can reach the point; so that it requires *an infinite number of parts of time*, but here the sophism quietly introduces *an infinite time* to catch the tortoise." De Morgan establishes the two convergent series, the one for the time, the other for the distance, passed over by Achilles, but he ignores the crucial question as to the reaching of the limit. In the article "Limit" he says that the variable "must never become equal" to its limit. Consequently De Morgan's exposition of limits, as given in these articles, lacked the generality necessary to explain the "Achilles."

On the Continent there prevailed the same diversity of definitions of a limit.

[1] D. Hume, *Essays Moral, Political, and Literary*, London, 1898, edited by T. H. Green and T. H. Grose, Vol. II, p. 129.

[2] William Whewell, *Doctrine of Limits*, Cambridge, 1838, p. 18. See also p. 23.

D'Alembert in 1754 puts no restriction upon the variable reaching its limit;[1] only, the variable must not "surpasser la grandeur dont elle approche." It is well known that there was a time when Lagrange was greatly troubled by the lack of rigor in the foundations of the calculus. He said:

> "That method [of limits] has the great inconvenience of considering quantities in the state in which they cease, so to speak, to be quantities; for though we can always well conceive the ratio of two quantities, as long as they remain finite, that ratio offers to the mind no clear and precise idea, as soon as its terms become, the one and the other, nothing at the same time."[2]

In the nineteenth century Carnot[3] and Cauchy[4] put no restriction upon variables reaching their limits. In 1817 Bolzano, whose writings did not at the time receive the attention they deserved, was concerned with the limits of continuous functions which attain their limits.[5] Later some French writers thought it necessary to impose restrictions. With Duhamel[6] the variable "never reaches" its limit. In Germany Klügel[7] gives a definition placing no restriction, but in the comments which follow the variable is pictured as not reaching its limit. In 1871 Hermann Hankel starts out in his article "Grenze"[8] by defining what is called a limit in mathematics; the limit is not reached. The difference between the variable and its limit he calls an infinitely small quantity—a quantity no multiple of which is capable of producing unity. But magnitudes of the same kind, by Euclid V, Def. 4, are such that some multiple of one will exceed the other. Hence an infinitely small line is not of the same kind as a finite one. This contradiction is to Hankel one of the indications that a scientific treatment of limits is still wanting. Hankel proceeds to express his adherence to the actual infinite and to develop a more satisfactory definition, free from restriction as to the attainment of the limiting value.

In the United States, as elsewhere, there has been great diversity of practice. Charles Davies of West Point, later of Columbia College, lets the variable reach its limit, in his *Calculus* of 1836. A discussion of this subject was carried on in the *Analyst* by Levi W. Meech, C. H. Judson, De Volson Wood and Simon Newcomb.[9] Wood's article voices the view that prohibiting the variable from attaining its limit "unnecessarily restricts the law of approach of the variable," though the variable can be "subjected to such a law that to the human mind it will appear impossible for it to reach the limit." An elaborate discussion of the

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[1] Article "Limite" in the *Encyclopédie, ou Dictionnaire raisonné des sciences* (Diderot).

[2] Quoted by Bledsoe, and by Carnot in his *Reflexions sur la métaphysique du calcul infinitesimal*, 5. éd., Paris, 1881, p. 147.

[3] Carnot, *op. cit.*, p. 168.

[4] A. L. Cauchy, *Cours d'analyse*, 1821, p. 4.

[5] Philip E. B. Jourdain, "The Development of the Theory of Transfinite Number," *Archiv der Mathematik u. Physik*, Bd. 14, 1909, p. 297. Jourdain's work appears in Bd. 10, 1906, pp. 254–281; Bd. 14, 1909, pp. 289–316; Bd. 16, 1910, pp. 21–43; Bd. 22, 1913.

[6] *Eléments de calcul infinitesimal*, Duhamel, Vol. I, Book. I, Chap. 1.

[7] G. S. Klügel, Mathematisches Wörterbuch, Leipzig, 1805, Vol. II, Art. "Grenze."

[8] *Allgem. Encyklopädie der Wissensch. u. Künste* (Brockhaus), 90. Theil.

[9] *The Analyst* (J. E. Hendricks, Des Moines, Iowa), Vol. I, 1874, p. 133 et seq.; Vol. VIII, 1881, p. 105 et seq.; Vol. IX, 1882, p. 79 et seq.; Vol. IX, 1882, p. 114 et seq.

subject is found in A. T. Bledsoe's *Philosophy of Mathematics*, Philadelphia, 1886. He holds (p. 44) that the variable never actually attains its limit, and

". . . this, I apprehend, will be found to be the case in relation to every variable really used in the infinitesimal method. It will, at least, be time enough to depart from the definition of Duhamel when variables are produced from the calculus which are seen to reach their limit without violating the law of their increase or decrease."

That a teacher who had pondered so long upon the foundations of the calculus as Bledsoe had done, could not think of examples of variables reaching their limits is an indication that the application of the calculus to physics and mechanics did not then receive the careful attention it deserved.

It is with the theory of limits as with negative numbers and imaginaries. In the eighteenth century it was felt that, whether such numbers could exist in algebra, was a matter of argument and demonstration; now it is merely a question of assumption. The same is true with variables reaching their limits. In modern theory it is not particularly a question of argument, but rather of assumption. The variable reaches its limit if we will that it shall; it does not reach its limit, if we will that it shall not. Our "willing" the one thing or the other consists in assuming a continuum in which the limit is a value the variable can assume; our "not willing" consists in not assuming, in the aggregate of values the variable can take, the value of the limit.

A GENERAL FORMULA FOR THE VALUATION OF BONDS.

By C. H. FORSYTH, University of Michigan.

It is the purpose of the present paper to generalize a formula for the valuation of bonds so that it will be applicable to a large number of bond offerings not satisfactorily covered by a known formula.

The problem to be considered may be put more concretely as follows. All formulas known up to the present time, cover the offering of bonds or loans only where the principal is repaid in *equal* installments. The case where payment of the principal is made in a lump sum is included, as a special case.

We shall derive a formula for computing the price of bonds where this principal is repaid in general in *unequal* installments. This formula will include cases where there may be only one installment of any one value. In fact, the formula will include as special cases not only the most general formula known at present but also all the special cases of the latter.

The most general formula[1] for valuation of bonds known up to this time is

$$k = \left(1 - \frac{a_{\overline{m(f+tr)}|} - a_{\overline{mf}|}}{r a_{\overline{mt}|}}\right)\left(\frac{g-i}{i}\right), \tag{1}$$

where k represents the premium or discount—as the case may be.

The nature of the above bond offering or loan is as follows. The principal

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[1] J. W. Glover, *A general formula for evaluating securities*, this MONTHLY, March, 1915.